Generation of jets on K3 surfaces

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March 28, 1998

Abstract

Let $L$ be an ample line bundle on a K3 surface $X$. We give sharp bounds on $n$ such that the global sections of $nL$ simultaneously generate $k$-jets on $X$.

1991 Mathematics Subject Classification. Primary 14J28, 14C20; Secondary 14C25, 14E25

Keywords and phrases. K3 surfaces, Seshadri constants, $k$-jet ampleness.

0. Introduction

Consider a K3 surface $X$ and an ample line bundle $L$ on $X$. It was established by Saint-Donat [7] – and follows also from Reider’s theorem [6] – that $\mathcal{O}_X(2L)$ is globally generated and $\mathcal{O}_X(3L)$ is very ample. The purpose of this note is to see how these basic facts generalize to the generation of jets and to jet ampleness. Recall that $L$ is said to generate $k$-jets at a point $x \in X$, if $L$ has global sections with arbitrarily prescribed $k$-jets at $x$, i.e. if the evaluation map

$$H^0(X, L) \to H^0 \left( X, L \otimes \mathcal{O}_X / \mathfrak{m}_x^{k+1} \right)$$

is surjective. A stronger variant of this local notion includes the separation of a finite set of distinct points: $L$ is $k$-jet ample, if for any choice of distinct points $x_1, \ldots, x_r$ in $X$ and positive integers $k_1, \ldots, k_r$ with $\sum_{i=1}^r k_i = k + 1$ the natural map

$$H^0(X, L) \to H^0 \left( X, L \otimes \mathcal{O}_X / \left( \mathfrak{m}_{x_1}^{k_1} \otimes \ldots \otimes \mathfrak{m}_{x_r}^{k_r} \right) \right)$$

is surjective (see [1]). This means that $L$ has global sections generating all simultaneous $k$-jets at any $r$ points $x_1, \ldots, x_r$.

Suitably high multiples of $L$ will certainly separate any given number of points and jets, so the interesting problem here is to determine optimal bounds. Certain effective statements on the local generation of jets, which are however not sharp, can be obtained by considering the Seshadri constant $\varepsilon(L, x)$, which measures the

*supported by DFG contract Ba 423/7-1.
†supported by the Europroj FLAG programme
‡supported by DFG grant 436 POL 113/75/0 and by KBN grant 2 PO3A 061 08
local positivity of $L$ at $x$. In fact, elementary arguments yield bounds for $\varepsilon(L, x)$ which, via vanishing, imply that the line bundle $\mathcal{O}_X(nL)$ generates $k$-jets at $x$ for $n \geq k + 2$ if $L$ is globally generated, and for $n \geq 2k + 4$ if the linear system $|L|$ has base points (see Sect. 3). Our main result gives the optimal bounds in this situation and also the sharp bound for jet ampleness:

**Theorem.** Let $X$ be a K3 surface, $L$ an ample line bundle on $X$ and $k$ a non-negative integer. Then either

(a) $\mathcal{O}_X(nL)$ is $k$-jet ample for $n \geq k + 2$, or

(b) $L$ is of the form $L = \mathcal{O}_X(aE + \Gamma)$, where $E \subset X$ is an elliptic curve, $\Gamma \subset X$ is a $(−2)$-curve with $E \cdot \Gamma = 1$ and $a \geq 3$.

In the exceptional case (b) let $\Delta$ be the finite set of singular points of the fibres of the elliptic fibration $X \rightarrow \mathbb{P}^1$ given by $|E|$. Then $\mathcal{O}_X(nL)$ generates $k$-jets at a point $x \in X - \Delta$ for $n \geq k + 2$ and it generates $k$-jets at a point $x \in \Delta$ if and only if $n \geq 2k + 1$.

**Notation and Conventions.** We work throughout over the field $\mathbb{C}$ of complex numbers.

For a $\mathbb{Q}$-divisor $D$ we denote by $\lceil D \rceil$ its round-up and by $\lfloor D \rfloor$ its round-down (integer part).

We will make use of the Kawamata-Viehweg vanishing theorem, which states that for a nef and big $\mathbb{Q}$-divisor $D$ on a smooth projective surface $S$ one has

$$H^i(S, \mathcal{O}_S(K_S + \lceil D \rceil)) = 0 \quad \text{for } i > 0.$$ 

Note that there is no normal crossing hypothesis in the surface case of Kawamata-Viehweg vanishing thanks to Sakai’s lemma (see [3, Lemma 1.1]).

Let $X$ be a non-singular surface. By a jet of order $k$ (or a $k$-jet) of a linear system $|L|$ on $X$ at the point $x \in X$ we mean an element $j \in H^0(X, L \otimes \mathcal{O}_X/m_x^{k+1})$. Accordingly, by a simultaneous jet of order $k$ (or a simultaneous $k$-jet) at the points $x_1, \ldots, x_r \in X$ we mean an element

$$j \in H^0(X, L \otimes \mathcal{O}_X/(m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r})) = \bigoplus_{i=1}^r H^0(X, L \otimes \mathcal{O}_X/m_{x_i}^{k_i}),$$

where $\sum_{i=1}^r k_i = k + 1$. Given a global section $s \in H^0(L)$, we consider the local Taylor expansions of $s$ around the $x_i$’s. A simultaneous $k$-jet is then given by the $\sum_{i=1}^r (k_i + 1)$-tuple of the coefficients of the terms of degree $\leq k_i$ for each $x_i$. In the proof of Lemma 2.1 we will use the term $0$-jet of order $k_i - 1$ at a point $x_i$ for the $(k_i + 1)$-tuple

$$0 = (0, \ldots, 0) \in H^0(X, L \otimes \mathcal{O}_X/m_{x_i}^{k_i}) = \mathbb{C}^{\binom{k_i+1}{2}}.$$
1. Linear systems with base points

Let $X$ be a K3 surface and $L$ an ample line bundle on $X$. We are interested in the jet ampleness of tensor powers $O_X(nL)$, $n \geq 1$. In this section we study the case where the linear system $|L|$ has base points. Under this assumption $L$ is of the form

$$L = O_X(aE + \Gamma),$$

(1)

where $E \subset X$ is an elliptic curve, $\Gamma \subset X$ is a smooth rational curve with $E \cdot \Gamma = 1$ and $a \geq 3$. (see [7, Proposition 8.1]). The $(-2)$-curve $\Gamma$ is the base locus of $|L|$. The pencil $|E|$ gives an elliptic fibration $X \to \mathbb{P}^1$. For $x \in X$ we will denote by $E_x$ the unique fibre passing through $x$. Because of $L \cdot E = 1$, all the fibres $E_x$ are irreducible. The singular fibres $E_x$ are rational curves with a single double point, which must lie outside $\Gamma$. Further, we will consider the finite set

$$\Delta = \{ x \in X \mid E_x \text{ is singular at } x \}$$

of the singular points of the fibres.

First, we show:

**Proposition 1.1** Suppose that $L$ is of the form (1) and let $x \in \Delta$. Then $O_X(nL)$ generates $k$-jets at $x$ for $n \geq 2k + 1$.

**Proof.** It is enough to show that

$$H^1\left(\tilde{X}, nf^*L - (k + 1)Z\right) = 0 \quad \text{for } n \geq 2k + 1,$$

(2)

where $f : \tilde{X} \to X$ is the blow-up of $X$ in $x$ and $Z = f^{-1}(x)$ is the exceptional divisor. Consider the divisor $D = aE_x + \Gamma \in |L|$ and the $\mathbb{Q}$-divisor

$$M = nf^*L - \left(k + \frac{1}{2}\right)Z - \lambda \left(nf^*D - \left(k + \frac{1}{2}\right)Z\right),$$

where $\lambda$ is defined as

$$\lambda = \frac{3}{n(4a - 1)}.$$

Since $\text{mult}_x(E_x) = 2$, one easily checks that the $\mathbb{Q}$-divisor $f^*L - \frac{1}{2}Z$ is nef and big, hence the numerical equivalence

$$M \equiv (1 - \lambda)\left(nf^*L - \left(k + \frac{1}{2}\right)Z\right)$$

shows that $M$ is nef and big for $n \geq 2k + 1$ as well. We will show that its round-up leads to the asserted vanishing.

Since $\text{mult}_x(D) = 2a$, we can write

$$f^*D = aE_x' + \Gamma' + 2aZ,$$
where $E'_x$ and $\Gamma'$ are the proper transforms of the curves $E_x$ respectively $\Gamma$. Therefore the round-up of $M$ is

$$[M] = nf^*L - \lfloor \lambda na \rfloor E'_x - \lfloor \lambda n \rfloor \Gamma' - \left[ 2\lambda na + (1 - \lambda) \left( k + \frac{1}{2} \right) \right] Z .$$

Our choice of $\lambda$ implies $\lambda na < 1$ and

$$2\lambda na + (1 - \lambda) \left( k + \frac{1}{2} \right) \geq k + 2 ,$$

hence we find

$$K_\tilde{X} + [M] = [M] + Z = nf^*L - (k + 1)Z - pZ$$

for some $p \geq 0$. The Kawamata-Viehweg vanishing theorem thus then gives (2), which in turn shows that $\mathcal{O}_X(nL)$ generates $k$-jets at $x$. \hfill $\square$

Now we prove that the bound $2k + 1$ in the previous proposition is, in fact, sharp:

**Proposition 1.2** Let $L$ be of the form (1) and $x \in \Delta$. Then $\mathcal{O}_X(nL)$ does not generate $k$-jets at $x$ if $n \leq 2k$.

**Proof.** First note that it is enough to prove the assertion for $n = 2k$, since $L$ is globally generated at $x$. Again let $f : \tilde{X} \to X$ be the blow-up at $x$, $Z$ the exceptional divisor and $D = \text{def} aE_x + \Gamma$.

From the exact sequence

$$0 \to \mathcal{O}_X(2kL) \otimes m_x^{k+1} \to \mathcal{O}_X(2kL) \to \mathcal{O}_X(2kL) \otimes \mathcal{O}_X/m_x^{k+1} \to 0$$

and $H^1(X, 2kL) = 0$ we see that it is sufficient to show that

$$H^1 \left( \tilde{X}, 2kf^*L - (k + 1)Z \right) \neq 0 .$$

Define

$$\lambda = \text{def} \frac{2}{k(2a - 1)}$$

and consider the $\mathbb{Q}$-divisor

$$M = \text{def} 2kf^*L - kZ - \lambda (2kf^*D - kZ) .$$

Because $\lambda < 1$ $M$ is certainly nef and big. Its round-up is

$$[M] = 2kf^*L - [2k\lambda a] E'_x - [2k\lambda] \Gamma' - [k + 4k\lambda a - k\lambda] Z ,$$

where as before $E'_x$ and $\Gamma'$ are the proper transforms. The main point in the construction of $M$ is that we have

$$1 < 2k\lambda a < 2, \ 0 < 2k\lambda < 1 \text{ and } k + 4k\lambda a - k\lambda = k + 2 ,$$
hence
\[ [M] = 2kf^*L - E'_x - (k + 2)Z, \]
which by Kawamata-Viehweg gives the vanishing
\[ H^i\left(\tilde{X}, 2kf^*L - E'_x - (k + 1)Z\right) = 0 \quad \text{for } i > 0. \]

The curve \( E'_x \) is the normalization of an irreducible singular elliptic curve, so it is smooth and rational. Consider the exact sequence

\[ 0 \rightarrow \mathcal{O}_{\tilde{X}} (2kf^*L - E'_x - (k + 1)Z) \rightarrow \mathcal{O}_{\tilde{X}} (2kf^*L - (k + 1)Z) \rightarrow \mathcal{O}_{E'_x} (2kf^*L - (k + 1)Z) \rightarrow 0. \]

From its associated long cohomology sequence (4) we get
\[ H^1\left(\tilde{X}, 2kf^*L - (k + 1)Z\right) \cong H^1\left(E'_x, 2kf^*L - (k + 1)Z\right). \]

But the restriction of \( 2kf^*L - (k + 1)Z \) to \( E'_x \) is of degree \(-2\), hence the right hand cohomology group in (5) does not vanish. This gives (3) and hence proves the proposition.

Next we show:

**Proposition 1.3** Suppose that \( L \) is of the form (1). Then \( \mathcal{O}_X(nL) \) generates \( k \)-jets at points \( x \in X - \Delta \) for \( n \geq k + 2 \).

**Proof.** Let \( x \) be a point in \( X - \Delta \) and let \( k \) be a non-negative integer. We again denote by \( f : \tilde{X} \rightarrow X \) the blow-up of \( X \) in \( x \) and by \( Z \) the corresponding exceptional divisor. Suppose \( n \geq k + 2 \). To prove the proposition, it is enough to show that
\[ H^1\left(\tilde{X}, nf^*L - (k + 1)Z\right) = 0. \]

First, we have
\[ (nf^*L - (k + 2)Z)^2 = n^2L^2 - (k + 2)^2 \geq (k + 2)^2L^2 - (k + 2)^2 > 0, \]

since the intersection pairing on \( X \) is even. Further, if \( C' \subset \tilde{X} \) is any irreducible curve different from the exceptional divisors, then we can write \( C' = f^*C - mZ \), where \( C \) is an irreducible curve on \( X \) and \( m = \text{mult}_x(C) \). We have
\[ (nf^*L - (k + 2)Z) \cdot C' = nL \cdot C - (k + 2)m. \]

If \( C \) is the base curve \( \Gamma \) or the fibre \( E_x \), then \( x \notin \Delta \) implies \( m \leq 1 \), so
\[ nL \cdot C - (k + 2)m \geq n - (k + 2) \geq 0. \]

If \( C \) is not as just described, the divisor
\[ D \overset{\text{def}}{=} (k + 1)(aE_x + \Gamma) \equiv (k + 1)L \]
meets $C$ properly, so

$$nL \cdot C \geq D \cdot C \geq a(k + 1)m + (k + 1)\Gamma \cdot C \geq (k + 2)m.$$  

So we have shown that $nf^*L - (k + 2)Z$ is nef and big, hence (6) follows from the Kawamata-Viehweg vanishing theorem, and we are done. \hfill \Box

**Remark 1.4** The bound $k + 2$ in the previous proposition is actually sharp. To see this, assume that $O_X(nL)$ generates $k$-jets on $X - \Delta$ for some $n \leq k + 1$ and consider the restriction of the bundle $O_X(nL)$ to a smooth elliptic fibre $E$. The restriction of $L$ to $E$ is of the form $O_E(p)$ for some point $p \in E$. Then $|O_E(nL)| = |O_E(np)|$ only generates $(n - 2)$-jets at $p$, since there is no meromorphic function on $E$ with a simple pole at $p$. A fortiori $|O_X(nL)|$ does not generate $(n - 1)$-jets at $p$.

2. Globally generated bundles

If a globally generated line bundle on a K3 surface fails to be very ample, then it gives a double covering of $\mathbb{P}^2$ or of a rational normal scroll. Therefore we begin this section by studying the generation of simultaneous $k$-jets in the set-up of a double covering.

**Lemma 2.1** Let $\pi : X \rightarrow Y$ be a double covering of smooth projective surfaces, branched over a smooth divisor $B \subset Y$. Suppose that $M \in \text{Pic}(Y)$ is a $k$-jet ample line bundle and that $O_Y(M - \frac{1}{2}B)$ is $(k - 1)$-jet ample. Then $L = \pi^*M$ is $k$-jet ample.

**Proof.** Let $R = \pi^*(B)_{\text{red}}$ and let $s_R \in H^0(R)$ be a section whose divisor of zeros is $R$. We observe that $s_R(-x) = -s_R(x)$ for all $x \in X$. The projection formula gives the following isomorphism

$$H^0(X, \pi^*M) = \pi^*H^0(Y, M) \oplus s_R \cdot \pi^*H^0\left(Y, M - \frac{1}{2}B\right).$$  

(7)

Now, let points $x_1, \ldots, x_r \in X$ and positive integers $k_1, \ldots, k_r$ with $\sum_{i=1}^r k_i = k + 1$ be given. Furthermore let

$$J \in H^0\left(X, \pi^*M \otimes O_X/\bigotimes_{i=1}^r m_{x_i}^{k_i}\right) = \bigoplus_{i=1}^r H^0\left(X, \pi^*M \otimes O_X/m_{x_i}^{k_i}\right)$$

be given. Let us write, corresponding to the above sum decomposition, $J = (j_1, \ldots, j_r)$, where $j_i$ is a $(k_i - 1)$-jet of $|\pi^*M|$ at $x_i$ for $i = 1, \ldots, r$. We can then write $J = \sum_{i=1}^r J_i$ where the simultaneous $k$-jets $J_i$ are of the form $J_i = (0, \ldots, 0, j_i, 0, \ldots, 0)$. In other words, $J_i$ has the 0-jet of order $k_i - 1$ as its $l$-th component, for $l \neq i$, and $j_i$ as its $i$-th component. It is enough to find for $i = 1, \ldots, r$ a section $s_i$ whose simultaneous $k$-jet at the points $x_1, \ldots, x_r$ is given
by $J_1$, since the sum $s = \sum_{i=1}^r s_i$ will then have the prescribed simultaneous jet $J$.
In order to alleviate notation, we assume $i = 1$.

We distinguish between three cases.

Case 1. Suppose that $x_1 \not\in R$ and that $x_2$ is the second point in the fibre of $\pi$ over $y_1 = \pi(x_1)$.

Let $p, q$ be local coordinates at the point $y_1 \in Y$. The pull-back of these coordinates gives rise to local coordinates $u_j, v_j$ around the points $x_j$ for $j = 1, 2$. In these local coordinates $j_1$ can be written as

$$j_1 = \sum_{i+j<k_1} a_{ij} u_i^1 v_j^1$$

(since we can set $a_{ij} = 0$ for $i + j \geq k_1$).

Let $\ell = \max(k_1, k_2)$. Since $M$ and $\mathcal{O}_Y (M - \frac{1}{2} B)$ are $(k-1)$-jet ample and $\ell + \sum_{i\geq 3} k_j \leq k$, there are sections $s \in H^0(Y, M)$ and $t \in H^0 (Y, M - \frac{1}{2} B)$ satisfying the following conditions

- $s \mod m^\ell_{y_1} = \frac{1}{2} \sum_{i+j<\ell} a_{ij} p^i q^j$,
- $s \mod m^k_{\pi(x_i)} = 0$ for $i \geq 3$,
- $(s_R \cdot \pi^* t) \mod m^\ell_{x_1} = \frac{1}{2} \sum_{i+j<\ell} a_{ij} u_1^i v_1^j$,
- $t \mod m^k_{\pi(x_i)} = 0$ for $i \geq 3$.

Then we have

$$(s_R \cdot \pi^* t) \mod m^\ell_{x_2} = -\frac{1}{2} \sum_{i+j<\ell} a_{ij} u_2^i v_2^j,$$

which in turn implies that

$$\begin{align*}
(\pi^* s + s_R \cdot \pi^* t) \mod m^\ell_{x_1} &= j_1, \\
(\pi^* s + s_R \cdot \pi^* t) \mod m^k_{x_i} &= 0 \quad \text{for } i \geq 2.
\end{align*}$$

then set the section $\pi^* s + s_R \cdot \pi^* t = s_1$, it has in fact the prescribed jet $J_1$.

Case 2. Suppose that $x_1 \not\in R$ and that the other point in the fibre of $\pi$ over $y_1 = \pi(x_1)$ is not among $x_2, \ldots, x_r$. Then, keeping the notation from the preceding case, the $k$-jet ampleness of $M$ implies that there exists a section $s \in H^0(Y, M)$ such that

- $s \mod m^k_{y_1} = \sum_{i+j<k_1} a_{ij} p^i q^j$,
- $s \mod m^k_{\pi(x_i)} = 0$.

Now $\pi^* s := s_1$ has the prescribed jet $J_1$. 
Case 3. We assume now that \( x_1 \in R \). Since \( B \) is smooth there are local coordinates \((p, q)\) at the point \( y_1 = \pi(x_1) \) such that \( B = \{ p = 0 \} \). The local coordinates \((u, v)\) around the point \( x_1 \) can be chosen in such a way that locally around \( x_1 \)
\[
\pi : (u, v) \mapsto (p = u^2, q = v).
\]
So we have \( R = \{ u = 0 \} \) locally around the point \( x_1 \). We can write the jet \( j_1 \) in the following way:
\[
j_1 = \sum_{2i+j<k_1} a_{2i,j} u^{2i} v^j + \sum_{2i+1+j<k_1} a_{2i+1,j} u^{2i+1} v^j
\]
\[
= \sum_{2i+j<k_1} a_{2i,j} u^{2i} v^j + u \cdot \sum_{2i+1+j<k_1} a_{2i+1,j} u^{2i+1} v^j.
\]
Since \( M \) is \( k \)-jet ample there exists a section \( s \in H^0(Y, M) \) satisfying
\[
\begin{align*}
\text{• } s &\mod m_{y_1}^{k_1} = \sum_{2i+j<k_1} a_{2i,j} p^i q^j, \\
\text{• } s &\mod m_{y_1}^{k_i} = 0 \text{ for } i \geq 2.
\end{align*}
\]
Similarly, since \( O_Y(M - \frac{1}{2} B) \) is \((k - 1)\)-jet ample there exists a section \( t \in H^0(Y, M - \frac{1}{2} B) \) such that
\[
\begin{align*}
\text{• } t &\mod m_{y_1}^{k_i-1} = \sum_{2i+j<k_1-1} a_{2i+1,j} p^i q^j, \\
\text{• } t &\mod m_{y_1}^{k_i} = 0 \text{ for } i \geq 2.
\end{align*}
\]
It is now easy to check that \( \pi^* s + s_R \pi^* t \) has the prescribed jets at the points \( x_1, \ldots, x_r \).

We now apply the lemma to show the following

**Proposition 2.2** Let \( X \) be a K3 surface and \( L \) be an ample globally generated line bundle on \( X \). Then \( O_X(nL) \) is \( k \)-jet ample for \( n \geq k + 2 \).

**Proof.** If \( L \) is already very ample, then clearly \( O_X(nL) \) is \( n \)-jet ample and we are done (see [1, Corollary 2.1]). On the other hand, if \( L \) fails to be very ample, then we are in one of the following two cases.

Case 1. \( L^2 = 2 \). Then Riemann-Roch implies \( h^0(X, L) = 3 \) and \( L \) induces a 2 : 1 mapping \( \pi : X \rightarrow \mathbb{P}^2 \), which is branched over a smooth sextic \( B \subset \mathbb{P}^2 \). Setting \( M = O_{\mathbb{P}^2}(n) \) we have \( nL = \pi^* M \) and
\[
O_{\mathbb{P}^2} \left( M - \frac{1}{2} B \right) = O_{\mathbb{P}^2}(n) \otimes O_{\mathbb{P}^2}(-3) = O_{\mathbb{P}^2}(n-3),
\]
hence the assumptions of the lemma are satisfied for \( n \geq k + 2 \) and we are done.

Case 2. \( L^2 \geq 4 \). Then Theorem 5.2 in [7] implies that there are two possibilities: either
(i) there exists a genus 2 curve \( C \subset X \) such that \( L = \mathcal{O}_X(2C) \), or

(ii) there exists an elliptic curve \( E \subset X \) with \( L \cdot E = 2 \).

In the first case, since \( C \) is irreducible and since there are no linear systems on K3 surfaces having isolated base points, \( \mathcal{O}_X(C) \) is globally generated. From Case 1 it follows that \( \mathcal{O}_X(nC) \) is \( k \)-jet ample for \( n \geq k + 2 \), and \( \mathcal{O}_X(nL) = \mathcal{O}_X(2nC) \) is, in this case, even \( (2k + 2) \)-jet ample.

In the second case, since \( L \) is ample, Proposition 5.7 of [7] implies that \( L \) gives a \( 2 : 1 \) mapping

\[
\pi : X \longrightarrow \pi(X) \subset \mathbb{P}^{p_a(L)},
\]

where \( \pi(X) \) is a rational normal scroll of degree \( p_a(L) - 1 \). It is well-known that the Picard group of \( \pi(X) \cong \mathbb{F}_r \) is generated by divisors \( E_0 \) and \( f \) which satisfy \( E_0^2 = -r, E_0 \cdot f = 1 \) and \( f^2 = 0 \). Since \( X \) is a K3 surface the projection formula yields that \( L \) is of the form \( L = \pi^*(E_0 + bf) \) for some \( b > r \) and the branch locus \( B \) satisfies \( \mathcal{O}_{\mathbb{F}_r}(B) = \mathcal{O}_{\mathbb{F}_r}(2(2E_0 + (2 + r)f)) \). Setting \( M = n(E_0 + bf) \) we see that \( M \) is \( k \)-jet ample and

\[
\mathcal{O}_{\mathbb{F}_r} \left( M - \frac{1}{2}B \right) = \mathcal{O}_{\mathbb{F}_r} \left( (n - 2)E_0 + (nb - 2 - r)f \right)
\]

is \( (k - 1) \)-jet ample for \( n \geq k + 2 \). Thus the assumptions of the lemma are satisfied and it yields our assertion.

The following example shows that the bound in the previous proposition is sharp.

**Example 2.3** Let \( \pi : X \longrightarrow \mathbb{P}^2 \) be a double cover branched over some smooth sextic \( B \subset \mathbb{P}^2 \) and let \( L = \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \). We are going to show that \( (k + 1)L \) is not \( k \)-jet ample. To this end, choose a point \( x \in R = \pi^*(B)_{\text{red}} \) and let \( u, v \) be local coordinates around \( x \) (as in Case 3 of the proof of Proposition 2.2).

Now consider the jet \( J \in H^0(X, \mathcal{O}_X((k + 1)L) \otimes \mathcal{O}_X/m_x^{k+1}) \), which is locally given as

\[
J = uv^{k-1}.
\]

If \( (k + 1)L \) were \( k \)-jet ample, then there would have to be a section \( s \) in \( H^0(X, (k + 1)L) \) such that \( s \mod m_x^{k+1} = J \). Equation (7) then shows that (locally) \( s \) is of the form

\[
s = \pi^*s' + u \cdot \pi^*s'',
\]

where \( s' \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k + 1)) \) and \( s'' \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k - 2)) \) are sections such that

\[
s'' \mod m_{\pi(x)} = v^{k-1},
\]

which is certainly impossible.
3. Seshadri constants on K3 surfaces

In this section we explain the relationship between our results and Seshadri constants of line bundles on K3 surfaces.

Recall that the Seshadri constant of a nef line bundle $L$ on a smooth projective variety $X$ at a point $x \in X$ is, by definition, the real number

$$\varepsilon(L, x) = \inf_{C \ni x} \frac{L \cdot C}{\text{mult}_x(C)},$$

where the infimum is taken over all irreducible curves $C \subset X$ passing through $x$ (see [2]). The number $\varepsilon(L, x)$ can be thought of as a measure of the local positivity of $L$ at the point $x$. The infimum

$$\varepsilon(L) = \inf_{x \in X} \varepsilon(L, x)$$

is the global Seshadri constant of $L$.

For K3 surfaces one has the following elementary observation:

**Proposition 3.1** Let $X$ be a K3 surface and let $L$ be an ample line bundle on $X$. If $L$ is globally generated, then we have

$$\varepsilon(L) \geq 1.$$  

If $L$ is not globally generated, then

$$\varepsilon(L) = \frac{1}{2}.$$  

**Proof.** The first part of the proposition holds on any smooth projective variety. It follows from the fact that if $|L|$ is free, then for any given curve $C \subset X$ and any point $x \in C$ there is a divisor in $|L|$ meeting $C$ properly at $x$.

If $L$ is of the form (1) and $C \subset X$ is singular at $x$, then either $L \cdot C/ \text{mult}_x(C) \geq 1$ or $C$ is a singular fibre of the elliptic fibration, in which case $L \cdot C/ \text{mult}_x(C) = 1/2$.

Via vanishing, bounds on $\varepsilon(L, x)$ yield criteria for the generation of jets: If $|L|$ is free, then $\varepsilon(L, x) \geq 1$ implies that $\mathcal{O}_X(nL)$ generates $k$-jets at $x$ for $n \geq k + 2$, and if $|L|$ has base points, then $\varepsilon(L, x) \geq \frac{1}{2}$ implies that $\mathcal{O}_X(nL)$ generates $k$-jets at $x$ for $n \geq 2k + 4$ (see [5, Proposition 5.7]).

**Remark 3.2** The proposition is also implied by the theorem stated in the introduction. This follows from the fact that the Seshadri constant of $L$ at $x$ is the relative number of jets that high multiples of $L$ generate at $x$:

$$\varepsilon(L, x) = \limsup_{n \to \infty} \frac{s(nL, x)}{n},$$

where $s(nL, x)$ is the maximal integer $s$ such that $\mathcal{O}_X(nL)$ generates $s$-jets at $x$ ([2, Theorem 6.4]).
References


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