Abstract

We present a variation of quasi-isometry to approach the problem of defining a geometric notion equivalent to commensurability. In short, this variation can be summarized as “quasi-isometry with uniform parameters for a large enough family of generating systems”. An introductory notion (based on isometries instead) is presented as well.

1 Basic Notions and Results

1.1 Quasi-Isometries

The first notion that can be vaguely identified as a sort of quasi-isometry dates back to 1941, when Stanislaw Ulam stated a question during a talk, whether linear functions are rigid. 32 years later, George Mostow introduced the notion of pseudo-isometry in his paper about rigidity of locally symmetric spaces (I), which then developed into today’s quasi-isometry. Another decade later, it became the main tool of Geometric Group Theory and through several landmark papers by Mikhail Gromov (e.g. II), the question was raised, to which extent the quasi-isometry class
of a group determines its algebraic structure. The area of Geometric Group Theory rapidly grew within the last 30 years, and though there have been brilliant advances in all directions, the classification of all finitely generated groups in terms of quasi-isometries still is uncompleted.

An overview about the ideas, methods, questions, articles and theorems of Geometric Group Theory is given in Pierre de la Harpe’s excellent “Topics in Geometric Group Theory” ([4]). A well written introduction to coarse geometry from a more general perspective is the book by D. Burago, Y. Burago and S. Ivanov ([3]).

In the following, let \( X \) and \( Y \) be metric spaces.

**Definition 1** Two (set theoretic) mappings \( \alpha, \beta : X \to Y \) are \( \epsilon \)-near to each other, \( \epsilon \geq 0 \), if
\[
d_Y(\alpha(x), \beta(x)) \leq \epsilon \quad \forall \ x \in X.
\]

A (set theoretic) mapping \( \alpha : X \to Y \) is \( \epsilon \)-surjective, \( \epsilon \geq 0 \), if for each \( y \in Y \) there is \( x \in X \) such that
\[
d_Y(\alpha(x), y) \leq \epsilon.
\]

**Definition 2** A (not necessarily continuous) map \( \eta : X \to Y \) is called a \((\lambda, \epsilon)\)-quasi-isometric embedding, \( \epsilon, \lambda \geq 0 \) (which shall always imply \( \lambda, \epsilon \in \mathbb{R} \)), if
\[
\lambda^{-1} d_X(x, x') - \epsilon \leq d_Y(\eta x, \eta x') \leq \lambda d_X(x, x') + \epsilon \tag{1}
\]
for all \( x, x' \in X \).

A pair \( \eta : X \to Y, \eta' : Y \to X \) of \((\lambda, \epsilon)\)-quasi-isometric embeddings is called a \((\lambda, \epsilon)\)-quasi-isometry if \( \eta \circ \eta' \) and \( \eta' \circ \eta \) are \( \epsilon \)-near the identities on \( Y \) and \( X \), respectively. When we speak of a “quasi-isometry \( \eta : X \to Y \)” a corresponding map \( \eta' \) shall always be implied.

\( X \) and \( Y \) are called quasi-isometric, if there is a quasi-isometry between them.

We say that two groups are quasi-isometric, if their word length metrics are quasi-isometric. The exact word length as well as the parameters \( \lambda \) and \( \epsilon \) of a quasi-isometry depend on the choice of a finite generating system, but the overall existence of a quasi-isometry is independent of the chosen system and hence the quasi-isometry class is a group invariant. It has been shown that several other group invariants, like the Hirsch rank of nilpotent groups (see e.g. [2] and [11]), are invariant under quasi-isometries, which gives rise to conclusions about the group’s algebraic structure.

### 1.2 Commensurability

Quasi-isometry encompasses the idea of two groups being *approximately isomorphic*, but quasi-isometries are not the only way to do this. Particularly the pure group-theoretic notion of commensurability rivals the quasi-isometry, and their interplay is still an interesting research problem, and the topic of this paper. In the following, we use the definitions given in [4].

**Definition 3** Let \( G \) and \( H \) be groups. \( G \) and \( H \) are commensurable when there exist subgroups \( G' \leq G \) and \( H' \leq H \) of finite index, such that \( G' \) and \( H' \) are isomorphic as group.
G and H are commensurable up to finite kernels if there exists a finite sequence of groups $\Gamma_1, \ldots, \Gamma_N$ and homomorphisms $h_0, \ldots, h_N$
\[
G \xrightarrow{h_0} \Gamma_1 \xleftarrow{h_1} \Gamma_2 \xrightarrow{h_2} \Gamma_3 \xleftarrow{h_3} \cdots \xrightarrow{h_{N-1}} \Gamma_N \xrightarrow{h_N} H
\]
with finite kernels and images of finite index.

One easily sees that commensurability always implies commensurability up to finite kernels, which in turn always implies quasi-isometry, given that both groups are finitely generated. We quote without proof the following Proposition from [4], IV.28.

**Proposition 4** Two residually finite groups are commensurable if and only if they are commensurable up to finite kernels.

**Proposition 5** Let $G$ and $H$ be f.g. groups, and let $\eta : G \to H$ be a homomorphism and quasi-isometry. Then $G$ and $H$ are commensurable up to finite kernels.

**Proof** The kernel of $\phi$ is finite, because it is the preimage of a finite subset of $H$. And the image $\phi(G)$ is a subgroup of $H$ of finite index: $\phi(G)$ is $\epsilon$-dense in $H$. Let $B$ be the $\epsilon$-ball around the identity in $H$, then each element $h \in H$ can be written as $b \cdot \phi(g)$ for some $b \in B$ and $g \in G$. With this, the number of cosets of $G/\phi(g)$ can be at most as large as $\#B$, and in particular, it is finite. □

There is a multitude of cases in which quasi-isometry implies commensurability (for example f.g. abelian groups (Bieberbach Theorem), certain types of Baumslag-Solitar groups in [6] and [13], abelian-by-cyclic groups in [7]) but also a plenty supply of counter-examples (e.g. Lamplighter groups in [5], or $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ with certain choices for $A \in \text{GL}(2, \mathbb{Z})$, see [3], IV.44 and IV.30, and [2]).

### 1.3 Results

Our approach will be to require the existence of a sufficiently large class of quasi-isometries with bounded parameters ($\lambda, \epsilon$) for a whole family of generating systems. Our two main results, Corollaries 29 and 41, can be summarized as follows. (Some of the technical details and exact definitions are missing in this summary.)

**Corollary 29** Let $G$ and $H$ be groups with families $\mathcal{S}_G$ and $\mathcal{S}_H$ of finite generating systems, such that for each pair $g, h \in G$ (or $H$) with $g \neq h^{-1}$ one has
\[
\sup_{S \in \mathcal{S}_G} ||g||_S < \infty \quad \text{and} \quad \sup_{S \in \mathcal{S}_G} ||h||_S = \infty \quad \text{or vice versa.} \quad (3)
\]

Let $\phi : G \to H$ be a quasi-isometry with uniform parameters for $\mathcal{S}_G$ and $\mathcal{S}_H$ in one direction. Assume $H$ is non-abelian, or of exponent 2. Assume further that a number of special relations are not fulfilled by $G$ and $H$. Then $\phi$ is a homomorphism and $H$ is isomorphic to the quotient of $G$ by the finite subgroup $\ker \phi \trianglelefteq G$. 

3
Corollary 41 Let $G$ and $H$ be groups with families $\mathcal{S}_G$ and $\mathcal{S}_H$ of finite generating systems, such that only a finite number of elements of $G$ and $H$ is at bounded distance from the neutral element throughout $\mathcal{S}_G$ and $\mathcal{S}_H$. Let $\eta: G \to H$ be a quasi-isometry with uniform parameters for $\mathcal{S}_G$ and $\mathcal{S}_H$ in both directions. Then $G$ and $H$ are commensurable up to finite kernels.

The author has not yet applied the results of this article to interesting examples. The examples given throughout this article and particularly in Section 3.7 are meant to demonstrate that the presented claims are not empty and to visualize their inner workings and interactions.

1.4 Notation

Denote with:

- $C_n$ the cyclic group of order $n$,
- $F_n$ the free group on $n$ generators,
- $\text{Cay}(G, S)$ the Cayley graph of the group $G$ relative to the generating system $S \subseteq G$,
- $\mathbb{N}$ the natural numbers without zero,
- $x \vee y$ the maximum of $x, y \in \mathbb{R}$,
- $\|x\|_S$ the word length of $x$ relative to the finite generating system $S$,
- $\#M$ the cardinality of the set $M$,
- $\phi'$ a coarse inverse of $\phi$, see Definition 2.

2 Shared Isometries

Definition 6 Consider $\epsilon \geq 0$, and let $G$ be a finitely generated group. Let $\mathcal{S}$ be a non-empty family of finite generating systems. We define the $\mathcal{S}$-uniform quasi-isometries by

$$(\lambda, \epsilon)\text{-Isom}_\mathcal{S}(G) := \{ \phi: G \to G \mid \forall S \in \mathcal{S}: \phi \text{ is a } (\lambda, \epsilon)\text{-qi rel. to } S \}$$
$$\text{Isom}_\mathcal{S}(G) := (1, 0)\text{-Isom}_\mathcal{S}(G)$$
$$\text{UQIsom}_\mathcal{S}(G) := \bigcup_{\lambda, \epsilon \geq 0} (\lambda, \epsilon)\text{-Isom}_\mathcal{S}(G)$$

We call the elements of $\text{Isom}_\mathcal{S}(G)$ $\mathcal{S}$-shared isometries of $G$ (cp. [7]). We further define

$$\epsilon\text{-Iden}_\mathcal{S}(G) := \{ \phi: G \to G \mid \forall S \in \mathcal{S}: \phi \text{ is } \epsilon\text{-near the identity} \}$$
$$\text{Iden}_\mathcal{S}(G) := \bigcup_{\epsilon \geq 0} \epsilon\text{-Iden}_\mathcal{S}(G).$$

These definitions are similar to the definition of the quasi-isometry group $\text{QI}(G)$ of a metric space or group (the calculation of $\text{QI}$ is very difficult in general, see for example [7]), and we find composition to be a group structure on $\text{UQIsom}_\mathcal{S}(G)$ after quotienting out $\text{Iden}_\mathcal{S}(G)$. The difference between the quasi-isometry group $\text{QI}(G)$ and $\text{UQIsom}_\mathcal{S}(G)/\text{Iden}_\mathcal{S}(G)$ seems to be subtle, as we just demand $\lambda$ and $\epsilon$ to be uniformly bounded for all word metrics in $\mathcal{S}$, but this difference can be enormous, if $\mathcal{S}$ is chosen
large enough. On the other hand, if \( \mathcal{S} \) comprises only a finite number of generating systems, \( \text{UQIsom}_\mathcal{S}(G)/\text{Iden}_\mathcal{S}(G) \) equals \( \text{QI}(G) \), independently of the exact choice of \( \mathcal{S} \). We will begin with the examination of \( \text{UQIsom}_\mathcal{S}(G) \) in Section 3 and for now concentrate on the nearly trivial case of \( \text{Isom}_\mathcal{S}(G) \). We start with a simple observation, which resulted from a discussion with Laurent Bartholdi and Martin Bridson during the 2007 winter school “Geometric Group Theory” in Göttingen:

**Theorem 7** (A) Let \( \mathcal{S} = \mathcal{S}_{\text{sym}} \) be the family of all, possibly asymmetric, finite generating systems of \( G \). Then \( \text{Isom}_\mathcal{S}(G) \) is isomorphic to \( G \) (using possibly asymmetric distance functions).

(B) Let \( G \) be a group with a finite, symmetric generating system \( S_0 \) such that the following hold:

1. There are no \( s_1, s_2, s_3 \in S_0 \) with \( s_1s_2 = s_3 \). (Minimality: easy to achieve.)
2. There are no \( s_1, s_2 \in S_0 \), \( s_1 \neq s_2^{-1} \), with \( s_1^2s_2^{-2} = e \).
3. There are no \( s_1, s_2 \in S_0 \), \( s_1 \neq s_2^{-1} \), with \( s_1^2 = s_2^{-1} \).
4. There are no \( s_1, s_2 \in S_0 \), \( s_1 \neq s_2^{-1} \), with \( s_1^2 = s_2 \) (In particular, \( G \) is not an abelian group.)
5. There are at least two distinct elements in \( S_0 \), which are not inverses of each other.

Let \( \mathcal{S} = \mathcal{S}_{\text{sym}} \) be the family of all symmetric finite generating systems of \( G \). Then \( \text{Isom}_\mathcal{S}(G) \) is isomorphic to \( G \).

(C) Let \( G \) be a f.g. abelian group without 2-torsion, and let \( \mathcal{S} = \mathcal{S}_{\text{sym}} \) be the family of all symmetric finite generating systems of \( G \). Then \( \text{Isom}_\mathcal{S}(G) \) is isomorphic to \( G \rtimes C_2 \), where \( C_2 \) acts by inversion \( x \mapsto x^{-1} \).

(D) Let \( G \) be a f.g. group, and \( S_0 \in \mathcal{S} = \mathcal{S}_{\text{sym}}(G) \), such that \( S_0 \) is minimal, and each element \( s \in S \) has order 2 (i.e. \( s^2 = e \)). Then \( \text{Isom}_\mathcal{S}(G) \cong G \).

**Proof** The proof is based on an idea by L. Bartholdi.

(A) Consider \( \phi \in \text{Isom}_\mathcal{S}(G) \), and \( x, s \in G \) arbitrary, \( s \neq e \). Let \( \mathcal{S}' := \{ S \in \mathcal{S} : s \in S \} \). Then \( d_S(x, xs) = 1 \) and \( d_S(\phi(x), \phi(xs)) = 1 \) for each \( S \in \mathcal{S}' \), i.e. \( s_x \in \mathcal{S}' \), \( \phi(x)^{-1} \cdot \phi(xs) \in \mathcal{S}' \). Assume \( s_x \neq s \). Then define \( S' := (S \setminus \{s_x\}) \cup \{s, s^{-1}s_x\} \). \( S' \) is again a generating system and \( s_x \notin S' \) as \( s \neq s_x \) and \( s \neq e \). Yet, we have \( s \in S' \), contradiction. So we conclude \( s_x = s \) and \( \phi(xs) = \phi(x) \cdot s \). By induction we find \( \phi(x) = \phi(e) \cdot x \), with \( \phi(e) \) arbitrary, i.e. \( \phi \) is a left-translation. On the other hand, each left-translation obviously is in \( \text{Isom}_\mathcal{S}(G) \), and

\[
G \ni g \mapsto (\phi_g : x \mapsto g \cdot x) \in \text{Isom}_\mathcal{S}(G)
\]

are shared isometries with \( \phi_g \circ \phi_h = \phi_{gh} \).

(B) Let \( \phi \in \text{Isom}_\mathcal{S}(G) \), and \( x \in G \) arbitrary, \( s \in S_0 \). Then \( d_{S_0}(x, xs) = 1 \) and \( d_{S_0}(\phi(x), \phi(xs)) = 1 \), i.e. \( s_x := \phi(x)^{-1} \cdot \phi(xs) \in S_0 \). Like in the asymmetric case, using \( \mathcal{S}' := \{ S \in \mathcal{S} : s \in S \} \) we find \( s_x = s \) or \( s_x = s^{-1} \), but the choice might depend on \( x \), and this is the main point differing from the asymmetric case. Now let \( r \in S_0 \) be arbitrary, \( r \neq s \) and \( S'_0 := S_0 \cup \{sr, (sr)^{-1}\} \). Note that \( d_{S_0}(x, xr) = 2 \), as there are no triangles in \( S_0 \), but \( d_{S'_0}(x, xsr) = 1 \). Let \( r_y = \phi(y)^{-1} \cdot \phi(yr) \in S_0 \), so we find \( \phi(xsr) = \phi(x) \cdot s_x \cdot r_{xs} \). As \( d_{S'_0}(\phi(x), \phi(xsr)) = 1 \), we have

1. \( s_x = s \) or \( s_x = s^{-1} \),
W e find eight cases:

1. \( s_x = s, \ r_x = r, \ s_x r_x = s r \)
2. \( s_x = s^{-1}, \ r_x = r, \ s_x r_x = s r \) \( \Rightarrow \) \( s^2 = e \land \text{case (1)} \)
3. \( s_x = s, \ r_x = r^{-1}, \ s_x r_x = s r \) \( \Rightarrow \) \( r^2 = e \land \text{case (1)} \)
4. \( s_x = s^{-1}, \ r_x = r^{-1}, \ s_x r_x = s r \) \( \Rightarrow \) \( s^2 r^2 = e \)
5. \( s_x = s, \ r_x = r, \ s_x r_x = r^{-1} s^{-1} \) \( \Rightarrow \) \( (sr)^2 = e \land \text{case (1)} \)
6. \( s_x = s^{-1}, \ r_x = r, \ s_x r_x = r^{-1} s^{-1} \) \( \Rightarrow \) \( r^s = r^{-1} \)
7. \( s_x = s, \ r_x = r^{-1}, \ s_x r_x = r^{-1} s^{-1} \) \( \Rightarrow \) \( s^r = s^{-1} \)
8. \( s_x = s^{-1}, \ r_x = r^{-1}, \ s_x r_x = r^{-1} s^{-1} \) \( \Rightarrow \) \( r^s = r \)

Cases (2), (3) and (5) directly lead to case (1) after re-inserting, case (4) contradicts property (2) for \( S_0 \), cases (6), (7) and (8) contradict properties (3) and (4). Hence, we are left with case (1), and \( s_x = s \) for all \( x \in G \).

Again, we use induction to show \( \phi(x) = \phi(e) \cdot x \), and get an isomorphism

\[
G \ni g \mapsto (\phi_0 : x \mapsto g \cdot x) \in \text{Isom}_G(G)
\]

(C) It is easy to find a generating system \( S_0 \) of \( G \) which fulfills all properties of subtheorem (B), except for property (4): \( s_1^{s_2} = s_1 \) is always true. We follow through the proof of subtheorem (B) until case (8) cannot be contradicted. Assume it is realized, i.e. we find \( x \in G, \ s \in S_0 \) with \( \phi(x) = \phi(e) \cdot x^{-1} \). Then, for each \( r \in S^{\pm 1} \setminus \{ s, s^{-1} \} \) we must have \( \phi(xr) = \phi(x) \cdot s^{-1} \cdot r^{-1} \), and from excluding all other cases and property (2) of \( S_0 \) we further find \( \phi(xs^2) = \phi(x) \cdot s^{-2} \). By induction and using the fact that \( S_0 \) generates \( G \), we show

\[
\phi(s_1 s_2 \ldots s_n) = \phi(e) \cdot s_1^{-1} s_2^{-1} \ldots s_n^{-1},
\]

or, due to abelianness, \( \phi(x) = \phi(e) \cdot x^{-1} \). Obviously, all these bijections are indeed shared isometries:

\[
d(\phi(x), \phi(y)) = d(x^{-1}, y^{-1}) = ||xy^{-1}|| \quad |\text{abelianness} \quad (7)
\]

\[
= ||y^{-1}x|| = d(y, x) \quad |\text{S}_0 \text{ is symmetric} \quad (8)
\]

\[
= d(x, y)
\]

Hence, we have \( \text{Isom}_G(G) \) isomorphic to \( G \times C_2 \) via

\[
G \times C_2 \ni (g, a) \mapsto (\phi_{(g, a)} : x \mapsto g \cdot x^a) \in \text{Isom}_G(G)
\]

(D) Once again, we follow through the proof of subtheorem (B). As \( S_0 \) is minimal, property (1) is automatically fulfilled. And as each \( s \in S_0 \) has order 2, the question \( s_x = s \) or \( s_x = s^{-1} \) is trivial, as \( s^{-1} = s \). Hence, we get the usual isomorphism

\[
G \ni g \mapsto (\phi_g : x \mapsto g \cdot x) \in \text{Isom}_G(G).
\]

\[
\square
\]

From now on, we will restrict to the symmetric case \( \mathfrak{S} = \mathfrak{S}_{\text{sym}} \).
Example 8 For groups $G$ with central elements it can be difficult to find a generating system $S_0$ satisfying the properties of Theorem $4B$, but typically it is still possible. Take for example:

$$G = \langle a, b, c \mid [a, c], [b, c] \rangle \cong (\mathbb{Z} \ast \mathbb{Z}) \times \mathbb{Z}$$

$$S_0 = \{a^{\pm 1}, (bc)^{\pm 1}, (ab)^{\pm 1}\}.$$  

Example 9 The same accounts for groups with 2-torsion. For example, because $C_2$ just as Theorem $7.D$ mentions; but not $C_2 \times C_2$, as one might think from Theorem $7.C$. Indeed, as inversion is the trivial operation in each group of exponent 2, we have $\text{Isom}_G(C_2) \cong (C_2)^n$ in the abelian case, contrary to Theorem $7.C$.

Example 10 For groups of the form $G = G_0 \times C_2$ with $C_2$ acting via inversion (written multiplicatively) on a f.g. group $G_0$ (which subsequently must be abelian), each element $(g, -1)$ with $g \in G_0$ has torsion 2. Given a minimal generating system $S_0$ of $G_0$, we can use

$$S := \{(g, -1) : g \in S_0\} \cup \{(e, -1)\}$$

to apply Theorem $4D$. And, just as it states, the inversion is not a shared isometry in this case: Let $G_0$ be any f.g. group with at least one element $s \in G_0$ with $s^2 \neq e$, $S_0$ a finite generating system of $G_0$ with $s \in S_0$, and $S' := S_0 \cup \{(s, -1)\}$, which generates $G = G_0 \times C_2$. Then holds $d((e, -1), (s, 1)) = 1$, as $(e, -1) \cdot (s, -1) = (s, 1)$, but

$$d((e, -1)^{-1}, (s, 1)^{-1}) = d((e, -1), (s^{-1}, 1)) > 1,$$

because $(s^{-1}, -1) \notin S'$. $(s \neq s^{-1}$, and $(s, -1)^{-1} = (s, -1).$)

Example 11 Similar to Example $4H$, consider a group $G = G_0 \rtimes H$, where a f.g. group $H$ acts on the f.g. abelian group $G_0$. The action shall be given by a non-trivial homomorphism $\alpha : H \to C_2$, where $C_2$ acts on $G_0$ by inversion. Furthermore, let $S_0$ be an arbitrary finite generating system of $G_0$, and let $S_H$ be a finite generating system for $H$, such that there are no two elements $s, t \in S$ with $s \neq t^\pm 1$ and $s^2 \neq s^{t^2} = e$. Finally, let $h_0 \in S_H$ be an element with $h_0^2 \neq e$. Then we can define a finite generating system

$$S_0 := S_H \cup \{gh_0 : g \in S_0\}$$

from which we choose a minimal subsystem $S \subseteq S_0$. Some simple calculations then show that the generating system $S$ fulfills the requirements for Theorem $4B$, and we conclude:

$$\text{Isom}_G(G_0 \rtimes H) \cong G_0 \rtimes H$$

In particular, this accounts for the group

$$\mathbb{Z} \rtimes \mathbb{Z} = \langle x, y : x^y = x^{-1} \rangle \cong \langle y, z : y^2 = z^2 \rangle.$$
It might seem strange to the reader that an automorphism of a group is not a shared isometry in general. As an example, we choose $G = F_2$ the free group on two generators $a$ and $b$ and its automorphism $\rho$ which maps $a$ to $b$ and vice versa. Now choose the generating system

$$S = \{a, b^2, b^3\}.$$  \hfill (20)

Then $\rho$ is not an isometry, because

$$||a||_S = 1,$$

but

$$||\rho(a)||_S = 2.$$  \hfill (21)

Considering the proof of Theorem 7 and the above examples, we are confident that the following statement can be proven just by application of more arduous combinatorics:

Let $G$ be a f.g. group, and let $\mathcal{S}$ be the family of all symmetric generating systems of $G$. Then $\text{Isom}_\mathcal{S}(G) \cong G \rtimes C_2$ if and only if $G$ is non-trivial, abelian, and not of exponent 2; $\text{Isom}_\mathcal{S}(G) \cong G$ otherwise.

Lemma 12 \textit{Let $G$, $H$ be f.g. groups, $\mathcal{S}_G$, $\mathcal{S}_H$ families of generating systems of $G$, $H$. Assume there is a bijection $\eta : G \to H$ such that}

- for each $S_G \in \mathcal{S}_G$ there is $S_H \in \mathcal{S}_H$ which makes $\eta : \text{Cay}(G, S_G) \to \text{Cay}(H, S_H)$ an isometry, and
- for each $S_H \in \mathcal{S}_H$ there is $S_G \in \mathcal{S}_G$ which makes $\eta^{-1} : \text{Cay}(H, S_H) \to \text{Cay}(G, S_G)$ an isometry.

\textit{Then $\text{Isom}_{\mathcal{S}_G}(G)$ and $\text{Isom}_{\mathcal{S}_H}(H)$ are isomorphic.}

\textbf{Proof} Define

$$\eta^* : \text{Isom}_{\mathcal{S}_G}(G) \to \text{Isom}_{\mathcal{S}_H}(H) \quad \phi \mapsto \eta \circ \phi \circ \eta^{-1}.$$  \hfill (22)

This is well-defined: For each $S_H \in \mathcal{S}_H$ choose $S_G \in \mathcal{S}_G$ such that $\eta$ is an isometry. Then $\eta \circ \phi \circ \eta^{-1} : H \to H$ is an isometry as well—vice versa for $(\eta^*)^{-1} := \eta^{-1} \circ \cdot \circ \eta$. Hence, $\eta^*$ is a bijection, and, as one easily computes, indeed an isomorphism between groups. \hfill $\square$

Corollary 13 \textit{Let $G$, $H$ and $\eta : G \to H$ be as in Lemma 12. Assume the requirements of Theorem 7, A, B, or D are met for both $G$ and $H$, or that $G$ and $H$ are both f.g. abelian without 2-torsion (case (C)). Then $G$ and $H$ are isomorphic.}

Sometimes it is possible to directly translate a proof into the rough context. This will be our goal for this article: To “roughificate” the proof of Theorem 8. For this goal, we have two ways to weaken the prerequisites of the theorem:

1. Switch from isometries to quasi-isometries.
2. Use less generating systems in a family than would be necessary to imply an isomorphism.
3 Uniform Quasi-Isometries

3.1 Definitions

In the following, we use the term “family” exclusively in the sense of “family of finite generating systems”.

Definition 14 Let $G$ be a f.g. group. We call a family $\mathcal{S}$ of finite generating systems of $G$ optimal if

$$\text{UQIsom}_{\mathcal{S}}(G) \cong \begin{cases} G \times C_2 & \text{if } G \text{ is abelian, non-trivial, and not} \\ G & \text{of exponent } 2, \end{cases}$$

(24)

Each translation from the left with an element of $G$ obviously is a uniform quasi-isometry (a shared isometry indeed), and for non-trivial abelian groups not of exponent 2, inversion is a uniform quasi-isometry as well, due to the symmetry of the distance function.

We will later see that adequate quasi-isometries between groups with optimal families still induce isomorphisms, so we will introduce families of less optimality.

Definition 15 Let $G$ be a group and $\mathcal{S}$ a family of finite generating systems of $G$. Let $g, h \in G$ be arbitrary. We say that $g$ and $h$ can be separated from each other by $\mathcal{S}$ if there is an $r \in \mathbb{N}$ such that for each $R \in \mathbb{N}$ there is $S = S(g, h, R) \in \mathcal{S}$ with $||g||_S \leq r$ and $||h||_S \geq R$, or vice versa ($||h||_S \leq r$ and $||g||_S \geq R$).

Equivalent definition: $g$ can be separated from $h$ if there is a subfamily $\mathcal{S}' \subseteq \mathcal{S}$ with

$$\sup_{S \in \mathcal{S}'} ||g||_S < \infty \text{ and } \sup_{S \in \mathcal{S}'} ||h||_S = \infty.$$ 

(25)

or vice versa.

Example 16 If $G$ or $\mathcal{S}$ is finite, no two elements can be separated from each other. Furthermore, an element and its inverse can never be separated from each other.

One easily sees from Definition 15 that two elements $g$ and $h \in G$ can not be separated from each other if and only if for each subfamily $\mathcal{S}' \subseteq \mathcal{S}$ we have

$$\sup_{S \in \mathcal{S}'} ||g||_S < \infty \text{ and only if } \sup_{S \in \mathcal{S}'} ||h||_S < \infty.$$ 

(26)

This shows that non-separability is an equivalence relation.

Definition 17 A dispersed group is a triple $(G, M, \mathcal{S})$ of a f.g. group $G$; a family $\mathcal{S}$ of finite generating systems of $G$; and an equivalence relation $M$ (seen as partition of $G$); such that the equivalence relation $M$ is the relation “$g$ and $h$ can not be separated in the sense of Definition 15”; $M(g)$ denotes the equivalence class of $g \in G$.

Example 18 Choose $G = \mathbb{Z}$ (additively written), $S_n := \{3, 3n + 1\}$ for any $n \in \mathbb{N}$, and $\mathcal{S} := \{S_n : n \in \mathbb{N}\}$. Then $M(0)$ is given by $3\mathbb{Z}$. The elements 3 and 1 clearly can be separated, 3 and 4 can be separated as well, but 1 and 4 can not be separated, neither 1 and 5. Actually, we find that there are two equivalence classes, $3\mathbb{Z}$ and $\mathbb{Z} \setminus 3\mathbb{Z}$. 

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Naturally, the partition $M$ is fully determined by $\mathcal{S}$. Yet it is an interesting question which partitions are possible for a given group $G$, i.e. how dispersible a group can be; and to what extent the family $\mathcal{S}$ is in turn determined by $M$. This is why we put $\mathcal{S}$ and $M$ on equal ranks in Definition 14.

The finest theoretically possible partition of a dispersed group obviously is given by equivalence classes of the form $M(g) := \{g, g^{-1}\}$, at least as long as we stick to symmetric distance functions (because $||g||_S = ||g^{-1}||_S$). We call this partition the optimal partition. We will later show that this optimal partition forces $\mathcal{S}$ to be an optimal family (at least for a large class of groups $G$), so this indeed is a specialization of Definition 14. On the other hand, there are groups which can not be optimally dispersed.

Now we introduce an enhanced notion of quasi-isometry what might be called “homomorphism” in a category of dispersed groups:

**Definition 19** Let $(G, M_G, \mathcal{S}_G)$ and $(H, M_H, \mathcal{S}_H)$ be dispersed groups. We call a pair of maps $\eta : G \rightarrow H$ and $\eta' : H \rightarrow G$ an $\mathcal{S}_G$-$\mathcal{S}_H$-uniform quasi-isometry if there are $\lambda, \epsilon \geq 0$ with:

- For each $S_H \in \mathcal{S}_H$ there is a finite generating system $S_G \in \mathcal{S}_G$ of $G$ which makes $(\eta, \eta') : \text{Cay}(G, S_G) \rightarrow \text{Cay}(H, S_H)$ a $(\lambda, \epsilon)$-quasi-isometry.

When we speak of an “$\mathcal{S}_G$-$\mathcal{S}_H$-uniform quasi-isometry $\eta : G \rightarrow H$” a suitable $\eta'$ shall always be implied.

Note that each $\mathcal{S}_G$-uniform quasi-isometry $\phi \in \text{UQIsom}_{\mathcal{S}_G}(G)$ (in the sense of Definition 14) is an $\mathcal{S}_G$-$\mathcal{S}_G$-uniform quasi-isometry (in the sense of Definition 19), but not the other way around: In the case of $\mathcal{S}_G$-$\mathcal{S}_G$-uniform quasi-isometries, for each $S_1 \in \mathcal{S}_G$ we might choose $S_2 \in \mathcal{S}_G$ such that $\phi$ is a quasi-isometry between the Cayley graphs $\text{Cay}(G, S_1)$ and $\text{Cay}(G, S_2)$, while in the case of an $\mathcal{S}_G$-uniform quasi-isometry, we have to use $S_2 = S_1$. This difference is crucial and will lead to a different and considerably more rigid behaviour of $\text{UQIsom}_{\mathcal{S}_G}(G)$ compared to the set of $\mathcal{S}_G$-$\mathcal{S}_G$-uniform quasi-isometries.

Finally we define the corresponding “isomorphism of dispersed groups”:

**Definition 20** Let $(G, M_G, \mathcal{S}_G)$ and $(H, M_H, \mathcal{S}_H)$ be dispersed groups. We call a pair of maps $\eta : G \rightarrow H$ and $\eta' : H \rightarrow G$ an $\mathcal{S}_G$-$\mathcal{S}_H$-bi-uniform quasi-isometry if $(\eta, \eta')$ is an $\mathcal{S}_G$-$\mathcal{S}_H$-uniform quasi-isometry and $(\eta', \eta)$ is an $\mathcal{S}_H$-$\mathcal{S}_G$-uniform quasi-isometry.

### 3.2 Uniform QIs are Almost Homomorphisms

**Lemma 21** Let $(G, M_G, \mathcal{S}_G)$ and $(H, M_H, \mathcal{S}_H)$ be dispersed groups and let $\phi : G \rightarrow H$ be an $\mathcal{S}_G$-$\mathcal{S}_H$-uniform quasi-isometry with $\phi(e) = e$.

Then $\phi$ fulfills:

\[ \forall g, h \in G \exists h' \in M(\phi(h)) : \phi(g \cdot h) = \phi(g) \cdot h'. \quad (27) \]

(Note that $h'$ depends on $g$ and $h$.)
**Proof** Let \( x, y \in G \) be arbitrary, \( z = y^{-1} x \), and \( z' = \phi(y)^{-1} \phi(x) \). Assume \( M(z') \neq M(\phi(z)) \). Then there is an \( r \in \mathbb{N} \) such that we may choose a suitable finite generating system

\[
S_H = S(z', \phi(z), (\lambda^2 r + 1) \cdot (\epsilon + 1)) \in \mathcal{G}_H
\]

(where \( \lambda \) and \( \epsilon \) are the parameters of \( \phi \) to separate \( z' \) from \( \phi(z) \)). We thus have

\[
||z'||_{S_H} = d_{S_H}(\phi(y), \phi(x)) \\
\leq \lambda \cdot d_{S_G}(y, x) + \epsilon \\
= \lambda \cdot d_{S_G}(\epsilon, z) + \epsilon \\
\leq \lambda^2 \cdot ||\phi(z)||_{S_H} + \lambda^2 \epsilon + \epsilon \\
= \lambda^2 \cdot \max(\lambda, 1) \cdot ||\phi(z)||_{S_H} + \lambda^2 \epsilon + \epsilon,
\]

and similarly:

\[
||\phi(z)||_{S_H} \leq \lambda^2 \cdot ||z'||_{S_H} + \lambda^2 \epsilon + \epsilon,
\]

Now one of \( ||z'||_{S_H} \) and \( ||\phi(z)||_{S_H} \) is \( \leq r \), while the other is larger than \( \lambda^2 r + \lambda^2 \epsilon + \epsilon \), contradiction. Hence, \( z' \) is in \( M(\phi(z)) \). Substituting \( y = g \) and \( z = h \) yields \( \phi(gh) = \phi(g) \cdot z' \) with \( z' \in M(\phi(h)) \).

Note that it is always possible to switch from an arbitrary uniform quasi-isomorphism \( \phi \) to one with \( \phi(\epsilon) = e \) by a simple left-translation. The left-translation even preserves the constants \( \lambda \) and \( \epsilon \) of the quasi-isometry.

For \( \mathcal{G}_G \)-uniform quasi-isomorphisms we find an even stronger result:

**Lemma 22** Let \((G, M, \mathcal{G})\) be a dispersed group and let \( \phi \in \text{UQI}(G) \) be arbitrary with \( \phi(\epsilon) = e \). Then \( \phi \) fulfills:

\[
\forall g, h \in G \exists h' \in M(h) : \phi(g \cdot h) = \phi(g) \cdot h'.
\]

(The difference to Lemma 21 is that \( M(\phi(h)) \) has been replaced by \( M(h) \).) As a special case \( (g = e) \), we find

\[
\forall h \in G : \phi(h) \in M(h).
\]

**Proof** The proof is very similar to the one of Lemma 21 but we assume \( M(z') \neq M(z) \) instead of \( M(\phi(z)) \) and apply the estimations \( ||\phi(z)||_{S} \leq \lambda ||z||_{S} + \epsilon \) and vice versa.

### 3.3 Optimal Families and Optimal Partitions

Lemmas 21 and 22 already show the first interesting direction: Choosing the partition \( M \) fine enough might turn the respective uniform quasi-isometries into full-fledged homomorphisms. The smallest possible partitions are the optimal ones. So our first questions will be: Are there optimal partitions? And do they yield optimal families? We will turn to the second question first.

**Lemma 23 (Optimality Lemma)** Let \((G, M, \mathcal{G})\) be a dispersed group with optimal partition \( M \).

Let \( G \) be such that \( x^y \neq x^{-1} \) for all \( x, y \in G \), unless \( x = x^{-1} \).

Then \( \mathcal{G} \) is optimal.
Proof Define \( \tilde{\phi}(g) := \phi(e)^{-1} \cdot \phi(g) \) for any \( g \in G \). Due to Lemma 24, and the optimal partition \( M \), we have
\[
\tilde{\phi}(gh) = \tilde{\phi}(g) \cdot h^{\pm 1}
\] (37)
for any \( g, h \in G \), and after multiplication with \( \phi(e) \), we see that we may drop the tilde. The sign might still depend on \( g \) and \( h \), we will get rid of this dependence in the next step.

Let \( c := \phi(e) \), and assume there are \( x, y \in G \) with \( \phi(x) = cx \neq cx^{-1} \), but \( \phi(xy) = c(xy)^{-1} \neq cx y \). Then
\[
c \cdot (xy)^{-1} = \phi(xy) = \phi(x) y^\alpha = cx y^\alpha
\] (38)
for some \( \alpha = \pm 1 \), hence \( x y^\alpha = y^{-1} x^{-1} \). If \( \alpha = +1 \), we have \((xy)^2 = e\), and hence \( \phi(xy) = c(xy) \). If \( \alpha = -1 \), we have \( x^y = x^{-1} \), which contradicts our premise, unless \( x = x^{-1} \). However, if \( x = x^{-1} \), we have \( \phi(x) = c x^{-1} \) as well.

We conclude that \( \phi(x) = cx \) for all \( x \in G \), or \( \phi(x) = cx^{-1} \) for all \( x \in G \). The latter case leads to
\[
c y x = \phi(x^{-1} y^{-1}) = \phi(x^{-1}) y^\beta = c x y^\beta
\] (39)
for all \( x, y \in G \), and some \( \beta = \pm 1 \). Again, the case \( \beta = -1 \) leads to \( y^\beta = y^{-1} \), which we excluded, unless \( y = y^{-1} \). So both cases for \( \beta \) lead to the conclusion that \( G \) must be abelian. Indeed, in the abelian case, the inversion is a shared isometry of all symmetric finite generating systems, and it is non-trivial if and only if \( G \) is not of exponent 2.

Hence, \( \text{UQIsom}_\phi(G) \cong G \times C_2 \) if and only if \( G \) is abelian, non-trivial, and not of exponent 2, \( \text{UQIsom}_\phi(G) \cong G \) otherwise. \( \square \)

Example 24 As already noted, no non-trivial finite group admits an optimal partition, as its diameter is limited. Torsion in itself is an obstruction to it: Assume \( G \) has an optimal partition, then each element of \( G \) is either torsionfree, or of exponent 2, 3, 4, or 6. These are the arguments for which the Euler totient function \( \varphi \) is 2 or less (14): Let \( x \in G \) be an element with \( x^n = e \). If \( \varphi(n) > 2 \), we can choose two different generators \( a, b \) of \( C_n \), and hence \( x^n \) and \( x^b \) are powers of each other (so they rapidly generate each other), yet \( (x^n) \neq (x^b)^{\pm 1} \). Still, there might be other optimal families for these groups.

Example 25 Let \( F_n \) be the free group generated by \( S_0 \) with \( \#S_0 = n \geq 2 \). Let \( g, h \in F_n \), \( g \neq h^{\pm 1} \), and \( R \in \mathbb{N} \) be arbitrary. Assume \( h \) is not a power of \( g \) and not neutral (otherwise switch them; both cannot happen as \( F_n \) is torsionfree). If \( g = e \), choose \( x \in S_0 \) such that \( h \) is not a power of \( x \), otherwise let \( x = g \). Let \( P \) be larger than \( R \) and larger than the word length of \( h \) in \( S_0 \). Define
\[
S(g, h, R) := \{ x \} \cup \{ x^{p^j} \ s_j \mid s_j \in S_0 \setminus \{x\}, j = 1, \ldots, \#(S_0 \setminus \{x\}) \} . \quad (40)
\]
The exponents \( p^j \) are chosen such that any non-trivial product of the elements \( x^{p^j} s_j \) has large enough word length in \( S_0 \), that it cannot equal \( h \), at least for the first \( R \) steps in the Cayley graph. After this, the powers
$x^p$ successively become available and “free” the generators $s_j$ to generate each remaining element of $F_n$, such that $||h||_S \geq R$. So we can separate $g$ from $h$ (even with $r = 1$) and the partition of the family $\mathcal{S}$ of all these generating systems is optimal. Lemma 28 now shows that $\mathcal{S}$ itself is optimal.

**Example 26** In a similar way, we may define optimal families for free abelian groups. We give the explicit example for $G = \mathbb{Z}$ (written additively): Again, assume $h$ is not zero, and let $P = 1 + (R \vee |h|)$, otherwise choose $P = 1 + (R \vee |h| \vee |g|)$ and $\phi$ must be a homomorphism. In a similar way, other prerequisites to sign is irrelevant and $\phi$ fulfills $\phi(gh) = \phi(g) \cdot \phi(h)^{\pm 1}$ for all $g, h \in G$, where the sign might depend on $g$ and $h$.

**Proof** This is a special case of Lemma 21.

In the case of Lemma 27 assume that $H$ is of exponent 2. Then the sign is irrelevant and $\phi$ must be a homomorphism. In a similar way, other prerequisites to $H$ can force $\phi$ to be a homomorphism.

**Theorem 28** Let $G, H$, and $\phi : G \to H$ be as in Lemma 26. Assume one of the following statements holds:

1. $G$ admits a generating system $S$ such that $(\phi(s))^2 = e$ for each $s \in S$.
2. $G$ admits a generating system $S$ such that:
   (a) There is no $x \in \phi(S \cup S^{-1})$, with $x^2 = e$.
   (b) There are no $x, y \in \phi(S \cup S^{-1})$, $x \neq y^{\pm 1}$, with $x^2 = y^2$.
   (c) There are no $x, y \in \phi(S \cup S^{-1})$, $x \neq y^{\pm 1}$, with $(xy)^2 = e$.
   (d) There are no $x, y \in \phi(S \cup S^{-1})$, $x \neq y^{\pm 1}$, with $x^b = x$.
   (e) There are no $x, y \in \phi(S \cup S^{-1})$, $x \neq y^{\pm 1}$, with $x^b = x^{-1}$.
   (f) There are at least two distinct elements in $S$, which are not inverses of each other.

(In particular, $G$ is not abelian.)

Then $G$ and $H$ are commensurable up to finite kernels (Definition 2).
Table 1: The sixteen cases of the proof of Theorem 28.3. For convenience, we use $x = \phi(s)$ and $y = \phi(t)$.

Proof Due to Lemma 27 we have in each case

$$\phi(g h) = \phi(g) \cdot \phi(h)^{\sigma(g, h)}$$

with $\sigma(g, h) \in \{\pm 1\}$ for any $g \in G$ and $h \in H$. Observe that $\sigma(g, e) = \sigma(e, g) = +1$. If $\phi(h)$ is neutral or of order 2, we choose $\sigma(g, h)$ to be +1 without loss of generality. We next show that under both hypotheses $\phi$ must be a homomorphism. Due to Proposition 5, $G$ and $H$ then must be commensurable up to finite kernels.

(1) We trivially have

$$\phi(g s) = \phi(g) \cdot \phi(s)$$

for any $g \in G$ and $s \in S$. By induction, $\phi$ must be a homomorphism.

(2) Let $g \in G$ and $s, t \in S$ be arbitrary, $s \not= t^{\pm 1}$. We make use of the associative law:

$$\phi(g s t) = \phi(g) \cdot \phi(s)^{\alpha} \cdot \phi(t)^{\beta}$$

$$= \phi(g) \cdot (\phi(s) \cdot \phi(t)^{\gamma})^{\delta}$$

for some $\alpha, \beta, \gamma, \delta = \pm 1$. The sixteen possible cases resolve as in Table 1. Fourteen cases subsequently contradict our premise. Both remaining cases 1 and 7 demand $\alpha = \sigma(g, s) = +1$, for all $g \in G$ and $s \in S$, so we have

$$\phi(g s) = \phi(g) \cdot \phi(s),$$

and, again by induction, $\phi$ must be a homomorphism. □
Corollary 29 Let $G$, $H$, and $\phi$ be as in Theorem 28 and let $H$ be non-abelian, or of exponent 2. In addition, $x^y \neq x^{-1}$ shall hold for all $x, y \in H$ with $x \neq x^{-1}$. Then $H$ is the quotient of $G$ by the finite subgroup $\ker \phi \leq G$.

Proof Lemma 28 ensures that $\phi \circ \phi' : H \to H$ is given by multiplication with a fixed element of $H$, and in particular, $\phi$ must be surjective. From the proof of Theorem 28 we know that $\phi$ is a homomorphism with finite kernel. Using the First Isomorphism Theorem ([1], Korollar 1.2.7), we see $H = \im \phi \cong G/\ker \phi$. □

3.4 Semi-direct Products

We will now turn to more general, non-optimal cases of dispersed groups. An example class of such groups is given by semi-direct products of dispersed groups with finite groups acting on them. Due to torsion, such groups can not admit an optimal partition.

Proposition 30 Let $(G, M_G, G)$ be a dispersed group and let $F$ be a finite group acting on $G$. Assume $G$ is compatible with the action, i.e. each $S_G \in G$ is closed under the action of $F$. (As a consequence, the partition is compatible as well, each $M_G$ is closed under the action of $F$.) Then their semi-direct product

$$(G \times F, M_{G \times F}, G_{G \times F})$$

(47)

with

$M_{G \times F}(g, t) := M_G(g) \times F \quad \forall (g, t) \in G \times F$  

(48)

and

$G_{G \times F} := \{(S_G \times F) \cup (\{e\} \times (F \setminus \{e\})) : S_G \in G\}$  

(49)

is a dispersed group as well.

Proof We have to show that two elements $(g, s), (h, t) \in G \times F$ are not $M_{G \times F}$-equivalent if and only if they can be separated by $G_{G \times F}$.

"⇒": By definition of $M_{G \times F}$, we have

$M_{G \times F}(g, s) \neq M_{G \times F}(h, t) \Leftrightarrow M_G(g) \neq M_G(h)$  

(50)

If this is the case, there is a generating system $S_G \in G$ which separates $g$ and $h$. Without loss of generality, assume $||g||_S_G \leq r$. We consider

$S_{G \times F} = (S_G \times F) \cup (\{e\} \times (F \setminus \{e\})) \in G_{G \times F}$.  

(51)

This set obviously is finite, generates $G \times F$, and fulfills $||g, s||_{S_{G \times F}} \leq r$. Furthermore, it is clear that $||(h, t)||_{S_{G \times F}} \leq ||h||_{S_G} + 1$. Now let

$$(h, e) = s_1 f_1 s_2 f_2 \ldots s_n f_n$$

(52)

be a presentation of $(h, e)$ in $S_{G \times F}$ such that $s_j$ or $f_j \neq e$ for each $j$; $(h, e)$ has a word length between $n$ and $2n$. However, as $S_G$ is closed under the action of $F$, we find that

$$h = s_1 s_2^{f_1} s_3^{f_2} s_4^{f_3} \ldots s_n^{f_{n-1}}$$

(53)

is a representation of $h$ with word length $\leq n$ as well. This gives the lower bound for $||(h, t)||_{S_{G \times F}}$. Hence we can separate $(g, s)$ from $(h, t)$ by $S_{G \times F}$.
Assume that \((g, s)\) and \((h, t)\) \(\in G\) can be separated from each other. Let \(R > 0\) be arbitrary and \(S_{G \times F} \in \mathfrak{S}_{G \times F}\) such that
\[
||(g, s)||_{S_{G \times F}} \leq r \quad \text{but} \quad ||(h, t)||_{S_{G \times F}} \geq R.
\] (54)

By definition, \(S_{G \times F}\) is of the form
\[
S_{G \times F} = (S_G \times F) \cup (\{e\} \times (F \setminus \{e\}))
\] (55)
for some \(S_G \in \mathfrak{S}_G\). Given shortest possible presentations of \(g\) and \(h\) in \(S_{G \times F}\), we are able to write \(g\) and \(h\) in terms of \(S_G\) (just as above), by which the word lengths change by a factor of \(\lambda = 2\) and an addend of \(\epsilon = 1\) at most. □

Example 31 As a special case of Proposition 30, we see that the direct product of a dispersed group and a finite group (with trivial action) is dispersed.

3.5 UQIsom / Iden

We start off with an analysis of the group \(\text{Iden}_\mathfrak{S}(G)\) of a dispersed group, which is connected to the subset \(M(e)\).

Proposition 32 Let \((G, M, \mathfrak{S})\) be a dispersed group.

1. Right-translation with \(g \in G\) is in \(\text{Iden}_\mathfrak{S}(G)\) if and only if \(g \in M(e)\).
2. Left-translation with \(g \in G\) is in \(\text{Iden}_\mathfrak{S}(G)\) if and only if all conjugates of \(g\) are in uniformly and \(\mathfrak{S}\)-uniformly bounded distance from \(e\).
3. If \(M(e)\) is trivial, only the trivial right- and left-translation is in \(\text{Iden}_\mathfrak{S}(G)\).
4. If left-translation with \(g \in G\) is in \(\text{Iden}_\mathfrak{S}(G)\), then so is conjugation with \(g\).

Proof (1) For each \(S \in \mathfrak{S}\) we have \(d_S(x, xg) = d_S(e, g)\). This distance is bounded if and only if \(g\) cannot be separated from \(e\), i.e. \(g \in M(e)\).

(2) We have:
\[
(x \mapsto g \cdot x) \in \text{Iden}_\mathfrak{S}(G) \iff \exists R \in \mathbb{N} \forall x \in G \forall S \in \mathfrak{S}: \quad d_S(x, gx) = d_S(e, g^x) < R
\] (57)

(3) In both cases, \(g\) must be an element of \(M(e)\).

(4) If left-translation with \(g\) is in \(\text{Iden}_\mathfrak{S}(G)\), then \(g \in M(e)\) and hence right-translation with \(g\) and \(g^{-1}\) is in \(\text{Iden}_\mathfrak{S}(G)\) as well, and hence conjugation. □

Theorem 33 Let \((G, M_G, \mathfrak{S}_G)\) and \((H, M_H, \mathfrak{S}_H)\) be dispersed groups. Let \(\eta: G \to H\) be an \(\mathfrak{S}_G\)-\(\mathfrak{S}_H\)-bi-uniform quasi-isometry. Then \(\text{UQIsom}_{\mathfrak{S}_G}(G) / \text{Iden}_{\mathfrak{S}_G}(G)\) and \(\text{UQIsom}_{\mathfrak{S}_H}(H) / \text{Iden}_{\mathfrak{S}_H}(H)\) are isomorphic as groups.
Proof Define
\[
\xi : \text{UQIsom}_{\mathcal{G}}(G) \to \text{UQIsom}_{\mathcal{H}}(H) \quad (58)
\]
\[
\phi \mapsto \eta \circ \phi \circ \eta'.
\quad (59)
\]
As \(\eta, \phi, \text{and} \ \eta'\) are uniformly quasi-isometric, so is \(\xi(\phi)\), hence \(\xi\) is well-defined. Next we see that \(\xi\) still is well-defined after quotienting out \(\text{Iden}_{\mathcal{G}}(G)\) and \(\text{Iden}_{\mathcal{H}}(H)\): Assume \(\phi_1, \phi_2 \in \text{UQIsom}_{\mathcal{G}}(G)\) fulfill
\[
d(\phi_1(\eta'(x)), \phi_2(\eta'(x))) \leq \lambda \cdot d(\eta(\phi_1(x)), \phi_2(\eta(x))) + \epsilon.
\quad (60)
\]
for some fixed \(\epsilon' \geq 0\) and any \(g \in G\). Then we have
\[
d(\eta \circ \phi_1 \circ \eta'(x), \eta \circ \phi_2 \circ \eta'(x)) \leq \lambda \cdot d(\phi_1(\eta'(x)), \phi_2(\eta'(x))) + \epsilon
\quad (61)
\]
Hence we have a map
\[
\xi : \text{UQIsom}_{\mathcal{G}}(G)/ \text{Iden}_{\mathcal{G}}(G) \to \text{UQIsom}_{\mathcal{H}}(H)/ \text{Iden}_{\mathcal{H}}(H) \quad (63)
\]
which further fulfills
\[
\xi(\phi_1) \circ \xi(\phi_2) = \eta \circ \phi_1 \circ \eta' \circ \eta \circ \phi_2 \circ \eta'.
\quad (64)
\]
As \(\eta' \circ \eta\) is \(\epsilon\)-near the identity, it vanishes modulo \(\text{Iden}_{\mathcal{H}}(H)\). Therefore, \(\xi\) is a homomorphism.

In just the same way we show that
\[
\xi' : \text{UQIsom}_{\mathcal{H}}(H) \to \text{UQIsom}_{\mathcal{G}}(G) \quad (65)
\]
\[
\psi \mapsto \eta' \circ \psi \circ \eta.
\quad (66)
\]
is a homomorphism, and also the inverse of \(\xi\) modulo \(\text{Iden}_{\mathcal{H}}(H)\) and vice versa. By this follows that \((\xi, \xi')\) is an isomorphism.

Corollary 34 Let \((G, M_G, \mathcal{G}_G)\) and \((H, M_H, \mathcal{G}_H)\) be dispersed groups with optimal partitions \(M_G\) and \(M_H\). Assume that \(G\) and \(H\) both are non-abelian or of exponent 2 or trivial, and that \(x^y \neq x^{-1}\) holds for all \(x, y \in G\), unless \(x = x^{-1}\).

If \(G\) and \(H\) are \(\mathcal{G}_G - \mathcal{G}_H\)-bi-uniformly quasi-isometric, then \(G\) and \(H\) are isomorphic as groups.

Proof Lemma 23 shows that \(\mathcal{G}_G\) and \(\mathcal{G}_H\) are optimal families, by Definition 14 this means
\[
\text{Iden}_{\mathcal{G}_G} \subseteq \text{UQIsom}_{\mathcal{G}_G}(G) = G_{\text{left}} \text{ or } G_{\text{left}} \rtimes C_2 \quad (67)
\]
and the same for \(H\). Non-abelianess excludes the case \(G_{\text{left}} \rtimes C_2\). Furthermore, from optimality of the partitions follows that \(M_G(e)\) and \(M_H(e)\) are trivial and by Proposition 32.3 implies that \(\text{Iden}_{\mathcal{G}_G}\) and \(\text{Iden}_{\mathcal{G}_H}\) are trivial, so we finally have by Theorem 33:
\[
G \cong G_{\text{left}} = \text{UQIsom}_{\mathcal{G}_G}(G)/\text{Iden}_{\mathcal{G}_G}(G) \quad (68)
\]
\[
\cong \text{UQIsom}_{\mathcal{H}_H}(H)/\text{Iden}_{\mathcal{H}_H}(H) = H_{\text{left}} \cong H. \quad (69)
\]
□
Similar theorems hold in the abelian and mixed case, yielding \( G \times C_2 \cong H \times C_2 \) and \( G \times C_2 \cong H \), respectively. As an alternative proof, one might apply Corollary 24 twice, but the way we used above demonstrates where exactly the several assumptions on \( G \) and \( H \) enter.

In view of Lemma 23, which allows for a wide range of optimally dispersed groups, we want to summarize:

\[
\text{Optimal + bi-uniformly quasi-isometric } \Rightarrow \text{often isomorphic.}
\]

As we are interested in the much weaker notion of commensurability instead of isomorphism, we see that there are two ways to weaken the prerequisites: removing “optimal” or removing “bi”. We have already seen in Corollary 29 that

\[
\text{Optimal + uniformly quasi-isometric } \Rightarrow \text{often } H \cong G / \ker \eta
\]

We will now investigate the generic bi-uniformly quasi-isometric case.

### 3.6 UQIsom / M

Controlling \( \text{Iden}_G(G) \) by means of the dispersedness structure is rather difficult, while \( \text{UQIsom}_G(G) \) poses less of a problem. So, if we cannot control \( \text{Iden}_G(G) \), we should remove it completely. We now try to use the equivalence relation \( M \) itself as divisor.

**Proposition 35** Let \( (G, M, S_G) \) be a dispersed group. Let \( g, h \in G \) and \( \phi \in \text{UQIsom}_{S_G}(G) \) be arbitrary. Then holds

\[
M_G(g) = M_G(h) \Rightarrow M_G(\phi(g)) = M_G(\phi(h)).
\]  
(70)

Let \( (H, M_H, S_H) \) be another dispersed group and let \( \eta : G \to H \) be an \( S_G-S_H \)-uniform quasi-isometry. Then holds

\[
M_G(g) = M_G(h) \Rightarrow M_H(\eta(g)) = M_H(\eta(h)).
\]  
(71)

If \( \eta \) is an \( S_G-S_H \)-bi-uniform quasi-isometry, the implication even is an equivalence.

Assume \( \eta(e) = e \). The special case \( h = e \) yields \( \eta(M_G(e)) \subseteq M_H(e) \).

**Proof** (1) Assume \( M_G(\phi(g)) \neq M_G(\phi(h)) \). Then we can separate \( \phi(g) \) from \( \phi(h) \): There is \( r \in \mathbb{N} \) such that for all \( R \in \mathbb{N} \) there is \( S \in S_G \) with (without loss of generality) \( ||\phi(g)||_S \leq r \) and \( ||\phi(h)||_S \geq R \). By Definition 6 there are \( \lambda, \epsilon \geq 0 \), such that \( \phi \) is a \( (\lambda, \epsilon) \)-quasi-isometry, hence \( ||g||_S \leq \lambda r + \epsilon \) and \( ||h||_S \geq \lambda^{-1} R - \epsilon \). This shows that \( g \) and \( h \) can be separated from each other by \( S_G \): Contradiction.

(2) Analog to (1).

**Proposition 36** Let \( (G, M, S) \) be a dispersed group.

Then \( M(e) \) is a subgroup of \( G \) and given by

\[
M(e) = \left\{ g \in G : \sup_{S \in S} ||g||_S < \infty \right\}.
\]  
(72)

**Proof** Equation 72 follows directly from the definitions. Let \( g, h \in M(e) \) be arbitrary, i.e. neither \( g \) nor \( h \) can be separated from \( e \) by means of \( S \).

Due to triangle inequality, we have

\[
d_S(e, gh) \leq d_S(e, g) + d_S(e, h),
\]  
(73)
thus \( g \cdot h \) is contained in \( M(e) \) as well. □

The remaining subsets \( M(g) \) can be the cosets of \( M(e) \), but may differ from them as well, see Example 45.

**Definition 37** Let \( (G, M, S) \) be a dispersed group. Define the equivalence relation
\[
\phi \sim \psi \quad \text{iff} \quad \forall x \in G : M(\phi(x)) = M(\psi(x)) \tag{74}
\]
for arbitrary \( \phi, \psi \in U\text{QIsom}_G(G) \). Denote the quotient group by
\[
U(G, S) := U\text{QIsom}_G(G) / \sim. \tag{75}
\]

**Proof** “\( \sim \)” obviously is an equivalence relation. We have to show that \( U(G, S) \) is well-defined and a group.

We first note that “\( \sim \)” is compatible with multiplication: Given \( \phi_1 \sim \phi_2 \) and \( \psi_1 \sim \psi_2 \) we have for any \( x \in G \).
\[
M(\psi_1(x)) = M(\psi_2(x)) \Rightarrow M(\phi_1 \circ \psi_1(x)) = M(\phi_2 \circ \psi_2(x)) \tag{76}
\]
by Proposition 35. Because of \( \phi_1 \sim \phi_2 \) we find
\[
M(\phi_1 \circ \psi_1(x)) = M(\phi_2 \circ \psi_2(x)) = M(\phi'_2 \circ \psi_2(x)), \tag{77}
\]
i.e. \( \phi_1 \circ \psi_1 \sim \phi_2 \circ \psi_2 \).

The identity is given by the class of \( \text{id}_G \), i.e. all maps
\[
\phi \in U\text{QIsom}_G(G) \quad \text{with} \quad \forall x \in G : M(\phi(x)) = M(x). \tag{78}
\]
Iden\( _G \) is a subset of these (otherwise \( x \) and \( \phi(x) \) could be separated), which means that for any \( \phi \in U\text{QIsom}_G(G) \) we have \( \phi \phi' \sim \phi' \phi \sim \text{id}_G \) and that \( U(G, S) \) is a quotient of \( U\text{QIsom}_G(G) / \text{Iden}_G(G) \). □

**Proposition 38** Let \( (G, M, S) \) be a dispersed group and let \( g \in G \) be arbitrary. Let \( \tau_g : G \rightarrow G, x \mapsto gx \) be the left-translation with \( g \). If \( \tau_g \sim \text{id}_G \), then \( g \in M(e) \).

**Proof** We have
\[
\tau_g \sim \text{id}_G \quad \Leftrightarrow \quad \forall x \in G : M(gx) = M(x). \tag{79}
\]
Insert \( x = e \). □

Actually, we have \( \tau_g \sim \tau_h \) if and only if \( gh^{-1} \in M(e) \), due to an argument similar to the one in Proposition 35. This can be used to show that “\( \sim \)” is compatible with the multiplication in \( G \) left as well.

**Theorem 39** Let \( (G, M_G, S_G) \) and \( (H, M_H, S_H) \) be dispersed groups and \( \eta : G \rightarrow H \) an \( S_G, S_H \)-bi-uniform quasi-isometry. Then \( U(G, S_G) \) and \( U(H, S_H) \) are isomorphic as groups.

**Proof** One can either use a proof analog to the one of Theorem 33, or use Theorem 33 itself and apply that \( \eta \) is compatible with the equivalence relation “\( \sim \)”, see Proposition 36. □
Theorem 40 Let \((G, M, \mathcal{S})\) be a dispersed group. Assume \(M(e)\) is finite. Then \(U(G, \mathcal{S})\) is a finite quotient of the left-translations \(G_{\text{left}} \cong G\).

Proof By Lemma 22 we have \(\phi(x) \in M(x)\) and \(\phi \sim \text{id}_G\). What remains are the left-translations \(G_{\text{left}}\), of which some might vanish modulo \(\sim\) as well. But for the left-translation with \(g \in G\) to vanish, \(g\) must be an element of \(M(e)\) (see Proposition 38), which is finite by assumption. Therefore

\[
[G_{\text{left}} : U(G, \mathcal{S})] = [G_{\text{left}} : G_{\text{left}} / \sim] \leq \#M(e). \quad (80)
\]

\[\square\]

Corollary 41 Let \((G, M_G, \mathcal{S}_G)\) and \((H, M_H, \mathcal{S}_H)\) be dispersed groups with finite \(M_G(e)\) and finite \(M_H(e)\). Let \(\eta : G \to H\) be an \(\mathcal{S}_G-\mathcal{S}_H\)-bi-uniform quasi-isometry. Then \(G\) and \(H\) are commensurable up to finite kernels.

3.7 Examples and Outlook

Example 42 Assume that the partition \(M\) of a dispersed group \((G, M, \mathcal{S})\) is optimal. By Theorem 40 \(U(G, \mathcal{S})\) is given by left-translations modulo \(\sim\). However, \(M(e)\) is trivial in the optimal case, so \(U(G, \mathcal{S})\) must be the whole of \(G_{\text{left}}\).

Example 43 We return to the case of a semi-direct product, see Proposition 30. Let \((G, M_G, \mathcal{S}_G)\) be a dispersed group and let \(F\) be a finite group acting on \(G\), such that \(\mathcal{S}_G\) and \(M_G\) are compatible with the action. We have seen that the semi-direct product

\[
(G \rtimes F, M_{G \rtimes F}, \mathcal{S}_{G \rtimes F}) \quad (81)
\]

with \(M_{G \rtimes F}(g, t) := M_G(g) \times F \forall (g, t) \in G \times F \quad (82)\)
and \(\mathcal{S}_{G \rtimes F} := \{(S_G \times F) \cup \{(e) \times (F \setminus \{e\})\} : S_G \in \mathcal{S}_G\} \quad (83)\)

is a dispersed group as well. Now define

\[
\eta : G \leftrightarrow G \rtimes F \quad (84)
\]

\[
g \mapsto (g, e) \quad (85)
\]

\[
\eta' : G \rtimes F \mapsto G \quad (86)
\]

\[
(g, s) \mapsto g. \quad (87)
\]

\((\eta, \eta')\) is an \(\mathcal{S}_G-\mathcal{S}_{G \rtimes F}\)-bi-uniform quasi-isometry, so \(G\) and \(G \rtimes F\) are commensurable up to finite kernels (this of course is self-evident, but meant as an example to demonstrate the mechanics of Corollary 41).

Example 44 Consider \(G = \mathbb{Z}\) (additively written) and \(\phi : G \to G\) the map which equals the identity except for \(\phi(1) = 0\). Assume \((G, M, \mathcal{S})\) is a dispersed group such that \(\phi\) is \(\mathcal{S}\)-uniform. This implies \(\phi(g) \in M(g)\) for all \(g \in \mathbb{Z}\), in particular we have \(0 \in M(1)\) and consequently \(1 \in M(0)\). As \(M(0)\) is a subgroup, \(M(0)\) contains the whole of \(\mathbb{Z}\); the partition is trivial. Therefore \(\phi\) can only be uniform in trivial cases.
Example 45 We have seen in Lemma 28 that an $\mathcal{S}$-uniform quasi-isometry $\phi$ of a dispersed group $(G, M, \mathcal{S})$ with $\phi(e) = e$ fulfills $\phi(g) \in M(g)$ for all $g \in G$. One might ask whether the inverse is true as well: If $\phi$ is any quasi-isometry, does $\phi(g) \in M(g)$ for all $g \in G$ imply that $\phi$ is $\mathcal{S}$-uniform? We present a counter-example to this.

Consider $G = \mathbb{Z}$ and $\phi : G \to G, g \mapsto 2g$. Denote the set of odd integers $> 2$ with $\mathbb{N}_{odd}$ and define

$$S_n^x := \{x, xn + 1\}, \quad \mathcal{S} := \{S_n^x : x \in \mathbb{N}_{odd}, n \in \mathbb{N}\}.$$  

(88)

Each subfamily $\{S_n^x\}_{n \in \mathbb{N}}$ separates a subgroup $M^x(0) = x\mathbb{Z} \triangleleft \mathbb{Z}$ from the remaining elements (cp. Example 38), so the partition $M(g)$ belonging to $\mathcal{S}$ is given by $\bigcap_{n \in \mathbb{N}_{odd}} M^x(g)$, which is $\{\pm g \cdot 2^k : k \in \mathbb{N}_0\}$. (This is an example for a family with trivial $M(e)$ but without optimal partition.) Obviously, $\phi$ fulfills $\phi(g) = 2g \in M(g)$.

However, the quasi-isometry $\phi$ is not $\mathcal{S}$-uniform: Choose $g \in G$ and some $S_n^x$ arbitrary. We find

$$d_{S_n^x}(0, g) = |\alpha| + |\beta|$$

(89)

with $g = \alpha \cdot x + \beta \cdot (xn + 1), \quad |\alpha| + |\beta|$ minimal.  

(90)

(The Chinese Remainder Theorem applies when $n$ is chosen appropriately; we might even restrict to such $n$.) For $\phi(g) = 2g$ we find that $2x$ and $2\beta$ is a solution, but there might easily be better solutions. Choose $g$ such that $2 \cdot ||g||$ is approximately $x \cdot (xn + 1)$. Then $||2g||$ becomes very small (of order 1). The necessary quasi-isometry-parameters $\lambda$ and $\epsilon$ have to grow with $n$ to compensate for this, hence $\phi$ is not $\mathcal{S}$-uniform.

In contrast to the above, $\phi$ actually is $\mathcal{S}_1\cdot\mathcal{S}_2$-uniform for appropriately chosen families $\mathcal{S}_1$ and $\mathcal{S}_2$ of $G = \mathbb{Z}$.

Example 46 Let $H$ be a subgroup of finite index in $G$ and $h \in H$ an element of infinite order. Let $S_H$ be a fixed finite generating system of $H$ and $S_0$ a (minimal and finite) subset of $G$, such that $S_H \cup S_0$ generates $G$. Define

$$S := S_H \cup \{h^{P_j} s_j : s_j \in S_0\}$$

(91)

for increasingly large $P_j \in \mathbb{N}$. Finally, set $\mathcal{S}$ to be the family of all these generating systems $S$.

Then $H$ obviously is a subgroup of

$$M_G(e) = \left\{ g \in G : \sup_{S \in \mathcal{S}} ||g||_S < \infty \right\},$$

(92)

and any bi-uniform quasi-isometry $G \to H$ maps $M_G(e)$ to the corresponding group $M_H(e)$. Within these groups, we may now define new dispersedness structures and apply e.g. Corollary 34 or Corollary 41 to demonstrate isomorphy or commensurability (up to finite kernels) of $M_G(e)$ and $M_H(e)$ and hence commensurability (up to finite kernels) of $G$ and $H$ themselves.

Example 47 Consider the free group $G := F_2$ of two generators $a$ and $b$ and the subgroup $H$ generated by $S_H := \{ab, ba, a^{-1}b\}$. The subgroup $H$ is of index 2 and isomorphic to $F_3$, its cosets in $G$ can be represented by $H$
and $a \cdot H$. Let $\eta : H \leftrightarrow G$, $\eta' : G \to H$ be a canonical quasi-isometry. We define

$$\mathcal{S}_G := \{\{ab, ba, a^{-1}b, (ab)^P a\} : P \in \mathbb{N}\},$$

and $\mathcal{S}_H := \{\{ab, ba, a^{-1}b\}\}$. (93) (94)

The family $\mathcal{S}_G$ splits $G$ into two (infinite) classes $M_G(e) = H$ and $M_G(a) = a \cdot H$. The family $\mathcal{S}_H$ is finite, its induced partition trivial. $\eta' : G \to H$ is $\mathcal{S}_G$-$\mathcal{S}_H$-uniform; this again is trivial because $\mathcal{S}_H$ is finite. We apply Proposition 35 to $\eta'$ and find that $\eta'$ can be restricted to $M_G(e) = H$ and $M_H(e) = H$, where it is the identity.

The preceding examples demonstrate how a suitable, and possibly repeated, choice of generating systems may help to understand the subgroup structure of a single group and the connection to other groups via quasi-isometries. On the other hand, if a certain quasi-isometry is given, one may ask about families of generating systems for which the quasi-isometry is uniform or bi-uniform. Large enough families will lead to small $M(e)$-subgroups, such that $U(G, \mathcal{S})$ is of interest. Small families might result in $M(e)$ being of finite index, such that the quasi-isometry can be restricted to $M(e)$ and the argument repeats.

**Open Question** There are several questions this article still leaves open:

1. Is there a torsion-free group without an optimal partition, or which does not admit an optimal generating family? Does an optimal family always imply an optimal partition? (p. 13)

2. What happens in the general case of two dispersed groups being uniformly instead of bi-uniformly quasi-isometric?

3. Is there an adequate characterization for a family $\mathcal{S}$ of finite generating systems from which follows that $M(e)$ is finite or a subgroup of finite index?

4. What is the exact cause that e.g. Lamplighter groups can be quasi-isometric but not commensurable to each other? Are there no suitable families of generating systems or is the canonical quasi-isometry not uniform?

5. Given an arbitrary quasi-isometry $\phi$, which is the finest possible dispersedness partition such that $\phi$ still is uniform?

6. Assume $G$ acts on some manifold or is a lattice of a Lie group. Are there any canonical restrictions to the possible families of generating systems of $G$, or does there even exist a single natural family?

**References**


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